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Semiclassical limit for nonlinear Schrödinger equations with electromagnetic fields

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Abstract

We study the existence of standing waves for a class of nonlinear Schrödinger equations in \mathbb{R}^n , with both an electric and a magnetic field. Under suitable non-degeneracy assumptions on the critical points of an auxiliary function related to the electric field, we prove the existence and the multiplicity of complex-valued solutions in the semiclassical limit. We show that, in the semiclassical limit, the presence of a magnetic field produces a phase in the complex wave, but it does not influence the location of peaks of the modulus of these waves.

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1. Introduction

Let us consider the nonlinear Schrödinger equation

$$ih \frac{\partial \psi}{\partial t} = \left(\frac{h}{i} \nabla - A(x) \right)^2 \psi + U(x) \psi - f(x, \psi), \quad x \in \mathbb{R}^n, \quad (1)$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ ($n \geq 2$). The function $\psi(x, t)$ takes on complex values, h is the Planck constant, i is the imaginary unit. Here $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes a magnetic potential and the Schrödinger operator is defined by

$$\left(\frac{h}{i} \nabla - A(x) \right)^2 \psi := -h^2 \Delta \psi - \frac{2h}{i} A \cdot \nabla \psi + |A|^2 \psi - \frac{h}{i} \psi \operatorname{div} A.$$

Actually, in general dimension $n \geq 2$, the magnetic field B is a 2-form where $B_{i,j} = \partial_j A_i - \partial_i A_j$; in the case $n = 3$, $B = \operatorname{curl} A$. The function $U: \mathbb{R}^n \rightarrow \mathbb{R}$ represents an electric potential. In the sequel, for the sake of simplicity, we limit ourselves to the particular case in which $f(x, t) = K(x)|t|^{p-1}t$, with $p > 1$ if $n = 2$ and $1 < p < (n+2)/(n-2)$ if $n \geq 3$.

It is now well known that the nonlinear Schrödinger equation (1) arises from a perturbation approximation for strongly nonlinear dispersive wave systems. Many papers are devoted to the nonlinear Schrödinger equation and its solitary wave solutions.

In this paper we seek for standing wave solutions to (1), namely waves of the form $\psi(x, t) = e^{-iEt} u(x)$ for some function $u: \mathbb{R}^n \rightarrow \mathbb{C}$. Substituting this *ansatz* into (1), and denoting for convenience $\varepsilon = h$, one is led to solve the complex equation in \mathbb{R}^n

$$\left(\frac{\varepsilon}{i} \nabla - A(x) \right)^2 u + (U(x) - E)u = K(x)|u|^{p-1}u. \quad (\text{NLS})$$

Renaming $V(x) + 1 = U(x) - E$, we assume from now on that $1 + V$ is strictly positive on the whole \mathbb{R}^n . Moreover, by an obvious change of variables, the problem becomes that of finding some function $u: \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$\left(\frac{\nabla}{i} - A(\varepsilon x) \right)^2 u + u + V(\varepsilon x)u = K(\varepsilon x)|u|^{p-1}u, \quad x \in \mathbb{R}^n. \quad (S_\varepsilon)$$

Concerning nonlinear Schrödinger equation with external magnetic field, we firstly quote a paper by Esteban and Lions [15], where concentrations and compactness arguments are applied to solve some minimization problems associated to (S_ε) under suitable assumptions on the magnetic field.

The purpose of this paper is to study the time-independent nonlinear Schrödinger equation (S_ε) in the semiclassical limit. This seems a very interesting problem since the Correspondence's Principle establishes that Classical Mechanics is, roughly speaking, contained in Quantum Mechanics. The mathematical transition

is obtained letting to zero the Planck constant ($\varepsilon \rightarrow 0$) and solutions $u(x)$ of (S_ε) which exist for small value of ε are usually referred as semi-classical ones (see [19]).

We remark that in the linear case, Helffer et al. in [18,20] have studied the asymptotic behavior of the eigenfunctions of the Schrödinger operators with magnetic fields in the semiclassical limit. Note that in these papers the *wells* of the Schrödinger operators with magnetic fields are the same as those without magnetic field, so that one doesn't 'see' the magnetic field in the definition of the *well*. See also [6] for generalization of the results by [20] for potentials which degenerate at infinity.

In the case $A = 0$ (no magnetic field), a recent extensive literature is devoted to study the time-independent nonlinear Schrödinger equation (S_ε) in the semi-classical limit. We shortly recall the main results in literature. The first paper is due to Floer and Weinstein which investigated the one-dimensional nonlinear Schrödinger equation (with $K(x) = 1$) and gave a description of the limit behavior of $u(x)$ as $\varepsilon \rightarrow 0$. Really they proved that if the potential V has a non-degenerate critical point, then $u(x)$ concentrates near this critical point, as $\varepsilon \rightarrow 0$.

Later, other authors proved that this problem is really local in nature and the presence of an isolated critical point of the potential V (in the case $K(x) = 1$) produces a semi-classical solution $u(x)$ of (S_ε) which concentrates near this point. Different approaches are used to cover different cases (see [2,13,22–24]). Moreover when V oscillates, the existence of multi-bumps solutions has also been studied in [4,11,14,17]. Furthermore multiplicity results are obtained in [9,10] for potentials V having a set of degenerate global minima and recently in [3], for potentials V having a set of critical points, not necessarily global minima.

A natural answer arises: *how does the presence of an external magnetic field influence the existence and the concentration behavior of standing wave solutions to (1) in the semi-classical limit?*

A first result in this direction is contained in [21] where Kurata has proved the existence of least energy solutions to (S_ε) for any $\varepsilon > 0$, under some assumptions linking the magnetic field $B = (B_{i,j})$ and the electric potential $V(x)$. The author also investigated the semi-classical limit of the found least energy solutions and showed a concentration phenomenon near global minima of the electric potential in the case $K(x) = 1$ and $|A|$ is small enough.

Recently in [8], Cingolani obtained a multiplicity result of semi-classical standing waves solutions to (S_ε) , relating the number of solutions to (S_ε) to the *richness* of a set M of global minima of an auxiliary function Λ defined by setting

$$\Lambda(x) = \frac{(1 + V(x))^\theta}{K(x)^{2/(p-1)}}, \quad \theta = \frac{p+1}{p-1} - \frac{n}{2}$$

(see (41) in Section 4 for details) depending on $V(x)$ and $K(x)$. We remark that, if $K(x) = 1$ for any $x \in \mathbb{R}^n$, global minima of Λ coincides with global minima of V . The variational approach, used in [8], allows to deal with unbounded potential V

and does not require assumptions on the magnetic field. However this approach works only near global minima of Λ .

In the present paper we deal with the more general case in which the auxiliary function Λ has a manifold M of stationary points, not necessarily global minima. For bounded magnetic potentials A , we are able to prove a multiplicity result of semi-classical standing waves of (S_ε) , following the new perturbation approach contained in the recent paper [3] due to Ambrosetti, Malchiodi, and Secchi.

Now we briefly describe the proof of the result. First of all, we highlight that solutions of (S_ε) naturally appear as *orbits*: in fact, Eq. (S_ε) is invariant under the multiplicative action of S^1 . Since there is no danger of confusion, we simply speak about solutions. The complex-valued solutions to (S_ε) are found near least energy solutions of the equation

$$\left(\frac{\nabla}{i} - A(\varepsilon\xi)\right)^2 u + u + V(\varepsilon\xi)u = K(\varepsilon\xi)|u|^{p-1}u, \quad (2)$$

where $\varepsilon\xi$ is in a neighborhood of M . The least energy of (2) have the form

$$\begin{aligned} z^{\varepsilon\xi, \sigma} : x \in \mathbb{R}^n \mapsto & e^{i\sigma + iA(\varepsilon\xi) \cdot x} \left(\frac{1 + V(\varepsilon\xi)}{K(\varepsilon\xi)} \right)^{1/(p-1)} \\ & \times U\left((1 + V(\varepsilon\xi))^{1/2}(x - \xi)\right), \end{aligned} \quad (3)$$

where $\varepsilon\xi$ belongs to M and $\sigma \in [0, 2\pi]$. As in [3], the proof relies on a suitable finite-dimensional reduction, and critical points of the Euler functional f_ε associated to problem (S_ε) are found near critical point of a finite-dimensional functional Φ_ε which is defined on a suitable neighborhood of M . This allows to use Lusternik–Schnirelman category in the case M is a set of local maxima or minima of Λ . We remark that the case of maxima cannot be handled by using direct variational arguments as in [8].

Moreover in the case M is a set of critical points non-degenerate in the sense of Bott (see [5]) we are able to prove the existence of (at least) cup long of M solutions concentrating near points of M . For the definition of the cup long, refer to Section 5.

Firstly we present a special case of our results.

Theorem 1.1. *Assume that*

- (K1) $K \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ is strictly positive and K'' is bounded;
- (V1) $V \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ satisfies $\inf_{x \in \mathbb{R}^n} (1 + V(x)) > 0$, and V'' is bounded;
- (A1) $A \in L^\infty(\mathbb{R}^n, \mathbb{R}^n) \cap C^1(\mathbb{R}^n, \mathbb{R}^n)$, and the Jacobian J_A of A is globally bounded in \mathbb{R}^n .

If the auxiliary function Λ has a non-degenerate critical point $x_0 \in \mathbb{R}^n$, then for $\varepsilon > 0$ small, the problem (S_ε) has at least a (orbit of) solution concentrating near x_0 .

Actually, we are able to prove the following generalization.

Theorem 1.2. *As in Theorem 1.1, assume again (K1), (V1) and (A1) hold.*

If the auxiliary function Λ has a smooth, compact, non-degenerate manifold of critical points M , then for $\varepsilon > 0$ small, the problem (S_ε) has at least $\ell(M)$ (orbits of) solutions concentrating near points of M .

Finally we point out that the presence of an external magnetic field produces a phase in the complex wave which depends on the value of A near M . Conversely the presence of A does not seem to influence the location of the peaks of the modulus of the complex wave. Although we will not deal with this problem, we believe that in order to have a local C^2 convergence of the solutions, some assumption about the smallness of the magnetic potential A should be added, as done in [21] for minima of V .

Finally we point out that Theorem 1.1 and Theorem 1.2 hold for problems involving more general nonlinearities. See Remark 5.1 in the last section.

Notation. 1. The complex conjugate of any number $z \in \mathbb{C}$ will be denoted by \bar{z} .

2. The real part of a number $z \in \mathbb{C}$ will be denoted by $\operatorname{Re} z$.

3. The ordinary inner product between two vectors $a, b \in \mathbb{R}^n$ will be denoted by $a \cdot b$.

4. From time to time, when no confusion can arise, we omit the symbol dx in integrals over \mathbb{R}^n .

5. C denotes a generic positive constant, which may vary inside a chain of inequalities.

6. We use the Landau symbols. For example $O(\varepsilon)$ is a generic function such that $\limsup_{\varepsilon \rightarrow 0} \frac{O(\varepsilon)}{\varepsilon} < \infty$, and $o(\varepsilon)$ is a function such that $\lim_{\varepsilon \rightarrow 0} \frac{o(\varepsilon)}{\varepsilon} = 0$.

2. The variational framework

We work in the real Hilbert space E obtained as the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm associated to the inner product

$$\langle u | v \rangle = \operatorname{Re} \int_{\mathbb{R}^n} \nabla u \cdot \overline{\nabla v} + u \bar{v}.$$

Solutions to (S_ε) are, under some conditions we are going to point out, critical points of the functional formally defined on E as

$$f_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(\left| \left(\frac{1}{i} \nabla - A(\varepsilon x) \right) u \right|^2 + |u|^2 + V(\varepsilon x) |u|^2 \right) dx \\ - \frac{1}{p+1} \int_{\mathbb{R}^n} K(\varepsilon x) |u|^{p+1} dx. \quad (4)$$

In what follows, we shall assume that the functions V , K and A satisfy the following assumptions:

- (K1) $K \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ is strictly positive and K'' is bounded;
 (V1) $V \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ satisfies $\inf_{x \in \mathbb{R}^n} (1 + V(x)) > 0$, and V'' is bounded;
 (A1) $A \in L^\infty(\mathbb{R}^n, \mathbb{R}^n) \cap C^1(\mathbb{R}^n, \mathbb{R}^n)$, and the Jacobian J_A of A is globally bounded in \mathbb{R}^n .

Indeed,

$$\int_{\mathbb{R}^n} \left(\left| \left(\frac{1}{i} \nabla - A(\varepsilon x) \right) u \right|^2 \right) dx \\ = \int_{\mathbb{R}^n} \left(|\nabla u|^2 + |A(\varepsilon x) u|^2 - 2 \operatorname{Re} \left(\frac{\nabla u}{i} \cdot A(\varepsilon x) \bar{u} \right) \right) dx,$$

and the last integral is finite thanks to the Cauchy–Schwartz inequality and the boundedness of A .

It follows that f_ε is actually well-defined on E .

In order to find possibly multiple critical points of (4), we follow the approach of [3]. In our context, we need to find complex-valued solutions, and so some further remarks are due.

Let $\xi \in \mathbb{R}^n$, which will be fixed suitably later on: we look for solutions to (S_ε) “close” to a particular solution of the equation

$$\left(\frac{\nabla}{i} - A(\varepsilon \xi) \right)^2 u + u + V(\varepsilon \xi) u = K(\varepsilon \xi) |u|^{p-1} u. \quad (5)$$

More precisely, we denote by $U_c : \mathbb{R}^n \rightarrow \mathbb{C}$ a least-energy solution to the scalar problem

$$-\Delta U_c + U_c + V(\varepsilon \xi) U_c = K(\varepsilon \xi) |U_c|^{p-1} U_c \quad \text{in } \mathbb{R}^n. \quad (6)$$

By energy comparison (see [21]), one has that

$$U_c(x) = e^{i\sigma} U^\xi(x - y_0)$$

for some choice of $\sigma \in [0, 2\pi]$ and $y_0 \in \mathbb{R}^n$, where $U^\xi : \mathbb{R}^n \rightarrow \mathbb{R}$ is the unique solution of the problem

$$\begin{cases} -\Delta U^\xi + U^\xi + V(\varepsilon\xi)U^\xi = K(\varepsilon\xi)|U^\xi|^{p-1}U^\xi, \\ U^\xi(0) = \max_{\mathbb{R}^n} U^\xi, \\ U^\xi > 0. \end{cases} \quad (7)$$

If U denotes the unique solution of

$$\begin{cases} -\Delta U + U = U^p & \text{in } \mathbb{R}^n, \\ U(0) = \max_{\mathbb{R}^n} U, \\ U > 0, \end{cases} \quad (8)$$

then some elementary computations prove that $U^\xi(x) = \alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)x)$, where

$$\alpha(\varepsilon\xi) = \left(\frac{1 + V(\varepsilon\xi)}{K(\varepsilon\xi)} \right)^{1/(p-1)},$$

$$\beta(\varepsilon\xi) = (1 + V(\varepsilon\xi))^{1/2}.$$

It is easy to show, by direct computation, that the function $u(x) = e^{iA(\varepsilon\xi) \cdot x} U_c(x)$ actually solves (5).

For $\xi \in \mathbb{R}^n$ and $\sigma \in [0, 2\pi]$, we set

$$z^{\varepsilon\xi, \sigma} : x \in \mathbb{R}^n \mapsto e^{i\sigma + iA(\varepsilon\xi) \cdot x} \alpha(\varepsilon\xi) U(\beta(\varepsilon\xi)(x - \xi)). \quad (9)$$

Sometimes, for convenience, we shall identify $[0, 2\pi]$ and $S^1 \subset \mathbb{C}$, through $\eta = e^{i\sigma}$.

Introduce now the functional $F^{\varepsilon\xi, \sigma} : E \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} F^{\varepsilon\xi, \sigma}(u) &= \frac{1}{2} \int_{\mathbb{R}^n} \left(\left| \left(\frac{\nabla u}{i} - A(\varepsilon\xi)u \right) \right|^2 + |u|^2 + V(\varepsilon\xi)|u|^2 \right) dx \\ &\quad - \frac{1}{p+1} \int_{\mathbb{R}^n} K(\varepsilon\xi)|u|^{p+1} dx, \end{aligned}$$

whose critical point correspond to solutions of (5).

The set

$$Z^\varepsilon = \{z^{\varepsilon\xi, \sigma} \mid \xi \in \mathbb{R}^n \wedge \sigma \in [0, 2\pi]\} \simeq S^1 \times \mathbb{R}^n$$

is a regular manifolds of critical points for the functional $F^{\varepsilon\xi, \sigma}$.

It follows from elementary differential geometry that

$$\begin{aligned} T_{z^{\varepsilon\xi, \eta}} Z^\varepsilon &= \text{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial \sigma} z^{\varepsilon\xi, \sigma}, \frac{\partial}{\partial \xi_1} z^{\varepsilon\xi, \sigma}, \dots, \frac{\partial}{\partial \xi_n} z^{\varepsilon\xi, \sigma} \right\} \\ &= \text{span}_{\mathbb{R}} \left\{ i z^{\varepsilon\xi, \sigma}, \frac{\partial}{\partial \xi_1} z^{\varepsilon\xi, \sigma}, \dots, \frac{\partial}{\partial \xi_n} z^{\varepsilon\xi, \sigma} \right\}, \end{aligned}$$

where we mean by the symbol $\text{span}_{\mathbb{R}}$ that all the linear combinations must have real coefficients.

We remark that, for $j = 1, \dots, n$,

$$\begin{aligned} & \frac{\partial}{\partial \xi_j} z^{\varepsilon \xi, \sigma} \\ &= -\frac{\partial}{\partial x_j} z^{\varepsilon \xi, \sigma} + O(\varepsilon |\nabla V(\varepsilon \xi)|) \\ & \quad + i\alpha(\varepsilon \xi) e^{iA(\varepsilon \xi) \cdot x + i\sigma} U(\beta(\varepsilon \xi)(x - \xi)) \left(\frac{\partial}{\partial \xi_j} (A(\varepsilon \xi) \cdot x) + A_j(\varepsilon \xi) \right) \\ &= -\frac{\partial}{\partial x_j} z^{\varepsilon \xi, \sigma} + O(\varepsilon |\nabla V(\varepsilon \xi)| + \varepsilon |J_A(\varepsilon \xi)|) + i z^{\varepsilon \xi, \sigma} A_j(\varepsilon \xi), \end{aligned}$$

so that

$$\frac{\partial}{\partial \xi_j} z^{\varepsilon \xi, \sigma} = -\frac{\partial}{\partial x_j} z^{\varepsilon \xi, \sigma} + i z^{\varepsilon \xi, \sigma} A_j(\varepsilon \xi) + O(\varepsilon).$$

Collecting these remarks, we get that any $\zeta \in T_{z^{\varepsilon \xi, \sigma}} Z^\varepsilon$ can be written as

$$\zeta = i \ell_1 z^{\varepsilon \xi, \sigma} + \sum_{j=2}^{n+1} \ell_j \frac{\partial}{\partial x_{j-1}} z^{\varepsilon \xi, \sigma} + O(\varepsilon)$$

for some real coefficients $\ell_1, \ell_2, \dots, \ell_{n+1}$.

The next lemma shows that $\nabla f_\varepsilon(z^{\varepsilon \xi, \sigma})$ gets small when $\varepsilon \rightarrow 0$.

Lemma 2.1. *For all $\xi \in \mathbb{R}^n$, all $\eta \in S^1$ and all $\varepsilon > 0$ small, one has that*

$$\begin{aligned} \|\nabla f_\varepsilon(z^{\varepsilon \xi, \sigma})\| &\leq C(\varepsilon |\nabla V(\varepsilon \xi)| + \varepsilon |\nabla K(\varepsilon \xi)| + \varepsilon |J_A(\varepsilon \xi)| \\ &\quad + \varepsilon |\operatorname{div} A(\varepsilon \xi)| + \varepsilon^2), \end{aligned}$$

for some constant $C > 0$.

Proof. From

$$\begin{aligned} f_\varepsilon(u) &= F^{\varepsilon \xi, \eta}(u) + \frac{1}{2} \int_{\mathbb{R}^n} \left(\left| \frac{\nabla u}{i} - A(\varepsilon x)u \right|^2 - \left| \frac{\nabla u}{i} - A(\varepsilon \xi)u \right|^2 \right) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)] u^2 \\ &\quad - \frac{1}{p+1} \int_{\mathbb{R}^n} [K(\varepsilon x) - K(\varepsilon \xi)] |u|^{p+1} \end{aligned} \tag{10}$$

and since $z^{\varepsilon \xi, \eta}$ is a critical point of $F^{\varepsilon \xi, \eta}$, one has (with $z = z^{\varepsilon \xi, \eta}$)

$$\begin{aligned}
\langle \nabla f_\varepsilon(z) \mid v \rangle &= \operatorname{Re} \int_{\mathbb{R}^n} \left(\frac{1}{i} \nabla - A(\varepsilon \xi) \right) z \cdot (A(\varepsilon \xi) - A(\varepsilon x)) \bar{v} \\
&\quad + \operatorname{Re} \int_{\mathbb{R}^n} (A(\varepsilon \xi) - A(\varepsilon x)) z \cdot \overline{\left(\frac{1}{i} \nabla - A(\varepsilon \xi) \right) v} \\
&\quad + \operatorname{Re} \int_{\mathbb{R}^n} (A(\varepsilon \xi) - A(\varepsilon x)) z \cdot (A(\varepsilon \xi) - A(\varepsilon x)) \bar{v} \\
&\quad + \operatorname{Re} \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi)) z \bar{v} \\
&\quad - \operatorname{Re} \int_{\mathbb{R}^n} (K(\varepsilon x) - K(\varepsilon \xi)) |z|^{p-2} z \bar{v} \\
&= \varepsilon \operatorname{Re} \int_{\mathbb{R}^n} \frac{1}{i} (\operatorname{div} A(\varepsilon x)) z \bar{v} \\
&\quad + 2 \operatorname{Re} \int_{\mathbb{R}^n} (A(\varepsilon \xi) - A(\varepsilon x)) z \cdot \overline{\left(\frac{1}{i} \nabla - A(\varepsilon \xi) \right) v} \\
&\quad + \operatorname{Re} \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi)) z \bar{v} \\
&\quad - \operatorname{Re} \int_{\mathbb{R}^n} (K(\varepsilon x) - K(\varepsilon \xi)) |z|^{p-2} z \bar{v}.
\end{aligned}$$

From the assumption that $|D^2 V(x)| \leq \text{const.}$ one infers

$$|V(\varepsilon x) - V(\varepsilon \xi)| \leq \varepsilon |\nabla V(\varepsilon \xi)| \cdot |x - \xi| + c_1 \varepsilon^2 |x - \xi|^2.$$

This implies

$$\begin{aligned}
&\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 z^{\varepsilon \xi, \sigma^2} \\
&\leq c_1 \varepsilon^2 |\nabla V(\varepsilon \xi)|^2 \int_{\mathbb{R}^n} |x - \xi|^2 z^2 (x - \xi) \\
&\quad + c_2 \varepsilon^4 \int_{\mathbb{R}^n} |x - \xi|^4 z^2 (x - \xi).
\end{aligned} \tag{11}$$

A direct calculation yields

$$\begin{aligned} \int_{\mathbb{R}^n} |x - \xi|^2 z^2(x - \xi) &= \alpha^2(\varepsilon\xi) \int_{\mathbb{R}^n} |y|^2 U^2(\beta(\varepsilon\xi)y) dy \\ &= \alpha(\varepsilon\xi)^2 \beta(\varepsilon\xi)^{-n-2} \int_{\mathbb{R}^n} |y'|^2 U^2(y') dy' \leq c_3. \end{aligned}$$

From this (and a similar calculation for the last integral in the above formula) one infers

$$\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|^2 |z^{\varepsilon\xi, \sigma}|^2 \leq c_4 \varepsilon^2 |\nabla V(\varepsilon\xi)|^2 + c_5 \varepsilon^4. \quad (12)$$

Of course, similar estimates hold for the terms involving K . It then follows that

$$\|\nabla f_\varepsilon(z^{\varepsilon\xi, \eta})\| \leq C(\varepsilon |\operatorname{div} A(\varepsilon\xi)| + \varepsilon |\nabla V(\varepsilon\xi)| + \varepsilon |J_A(\varepsilon\xi)| + \varepsilon^2),$$

and the lemma is proved. \square

3. The invertibility of $D^2 f_\varepsilon$ on $(TZ^\varepsilon)^\perp$

To apply the perturbative method, we need to exploit some non-degeneracy properties of the solution $z^{\varepsilon\xi, \sigma}$ as a critical point of $F^{\varepsilon\xi, \sigma}$.

Let $L_{\varepsilon, \sigma, \xi} : (T_{z^{\varepsilon\xi, \sigma}} Z^\varepsilon)^\perp \rightarrow (T_{z^{\varepsilon\xi, \sigma}} Z)^\perp$ be the operator defined by

$$\langle L_{\varepsilon, \sigma, \xi} v \mid w \rangle = D^2 f_\varepsilon(z^{\varepsilon\xi, \sigma})(v, w)$$

for all $v, w \in (T_{z^{\varepsilon\xi, \sigma}} Z^\varepsilon)^\perp$.

The following elementary result will play a fundamental role in the present section.

Lemma 3.1. *Let $M \subset \mathbb{R}^n$ be a bounded set. Then there exists a constant $C > 0$ such that for all $\xi \in M$ one has*

$$\int_{\mathbb{R}^n} \left| \left(\frac{\nabla}{i} - A(\xi) \right) u \right|^2 + |u|^2 \geq C \int_{\mathbb{R}^n} (|\nabla u|^2 + |u|^2) \quad \forall u \in E. \quad (13)$$

Proof. To get a contradiction, we assume on the contrary the existence of a sequence $\{\xi_n\}$ in M and a sequence $\{u_n\}$ in E such that $\|u_n\|_E = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow +\infty} \left[\int_{\mathbb{R}^n} \left| \left(\frac{\nabla}{i} - A(\xi) \right) u_n \right|^2 + \int_{\mathbb{R}^n} |u_n|^2 \right] = 0. \quad (14)$$

In particular, $u_n \rightarrow 0$ strongly in $L^2(\mathbb{R}^n, \mathbb{C})$. Moreover, since M is bounded, we can assume also $\xi_n \rightarrow \xi^* \in \bar{M}$ as $n \rightarrow \infty$. From

$$\begin{aligned} & \int_{\mathbb{R}^n} \left| \left(\frac{\nabla}{i} - A(\xi_n) \right) u_n \right|^2 \\ &= \int_{\mathbb{R}^n} \left(|\nabla u_n|^2 + |A(\xi_n)|^2 |u_n|^2 - 2 \operatorname{Re} \frac{1}{i} \nabla u_n \cdot A(\xi_n) \overline{u_n} \right) \end{aligned}$$

we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^n} |\nabla u_n|^2 &= 1, \\ \lim_{n \rightarrow +\infty} \operatorname{Re} \int_{\mathbb{R}^n} \frac{1}{i} \nabla u_n \cdot A(\xi_n) \overline{u_n} &= \frac{1}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^n} |\nabla u_n| |A(\xi_n)| |u_n| &\geq \limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^n} \frac{1}{i} \nabla u_n \cdot A(\xi_n) \overline{u_n} \right| \\ &\geq \limsup_{n \rightarrow \infty} \operatorname{Re} \int_{\mathbb{R}^n} \frac{1}{i} \nabla u_n \cdot A(\xi_n) \overline{u_n} = \frac{1}{2}. \end{aligned}$$

From this we conclude that

$$\begin{aligned} \frac{1}{2} &\leq |A(\xi^*)| \limsup_{n \rightarrow \infty} \|\nabla u_n\|_{L^2} \|u_n\|_{L^2} \\ &\leq |A(\xi^*)| \limsup_{n \rightarrow \infty} \|u_n\|_{L^2} = 0, \end{aligned}$$

which is clearly absurd. This completes the proof of the lemma. \square

At this point we shall prove the following result:

Lemma 3.2. *Given $\bar{\xi} > 0$, there exists $C > 0$ such that for ε small enough one has*

$$|(L_{\varepsilon, \sigma, \xi} v \mid v)| \geq C \|v\|^2, \quad \forall |\xi| \leq \bar{\xi}, \quad \forall \sigma \in [0, 2\pi], \quad \forall v \in (T_{z^{\varepsilon \xi, \sigma}} Z^\varepsilon)^\perp. \quad (15)$$

Proof. We follow the arguments in [3], with some minor modifications due to the presence of A . Recall that

$$T_{z^{\varepsilon \xi, \sigma}} Z = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial \xi_1} z^{\varepsilon \xi, \sigma}, \dots, \frac{\partial}{\partial \xi_n} z^{\varepsilon \xi, \sigma}, i z^{\varepsilon \xi, \sigma} \right\}.$$

Define

$$\mathcal{V} = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1} z^{\varepsilon \xi, \sigma}, \dots, \frac{\partial}{\partial x_n} z^{\varepsilon \xi, \sigma}, z^{\varepsilon \xi, \sigma}, i z^{\varepsilon \xi, \sigma} \right\}.$$

As in [3], it suffices to prove (15) for all $v \in \text{span}_{\mathbb{R}}\{z^{\varepsilon\xi,\sigma}, \phi\}$, where $\phi \perp \mathcal{V}$. More precisely, we shall prove that for some constants $C_1 > 0$, $C_2 > 0$, for all ε small enough and all $|\xi| \leq \bar{\xi}$ the following hold:

$$\langle L_{\varepsilon,\sigma,\xi} z^{\varepsilon\xi,\sigma} \mid z^{\varepsilon\xi,\sigma} \rangle \leq -C_1 < 0, \quad (16)$$

$$\langle L_{\varepsilon,\sigma,\xi} \phi \mid \phi \rangle \geq C_2 \|\phi\|^2 \quad \forall \phi \perp \mathcal{V}. \quad (17)$$

For the reader's convenience, we reproduce here the expression for the second derivative of $F^{\varepsilon\xi,\sigma}$:

$$\begin{aligned} D^2 F^{\varepsilon\xi,\sigma}(u)(v, v) &= \int_{\mathbb{R}^n} \left| \left(\frac{\nabla}{i} - A(\varepsilon\xi) \right) v \right|^2 + |v|^2 + V(\varepsilon\xi)|v|^2 \\ &\quad - K(\varepsilon\xi) \left[(p-1) \text{Re} \int_{\mathbb{R}^n} |u|^{p-3} \text{Re}(u\bar{v})u\bar{v} + \int_{\mathbb{R}^n} |u|^{p-1}|v|^2 \right]. \end{aligned}$$

Moreover, since $z^{\varepsilon\xi,\sigma}$ is a solution of (5), we immediately get

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\left| \left(\frac{\nabla}{i} - A(\varepsilon\xi) \right) z^{\varepsilon\xi,\sigma} \right|^2 + V(\varepsilon\xi)|z^{\varepsilon\xi,\sigma}|^2 + |z^{\varepsilon\xi,\sigma}|^2 \right) \\ = K(\varepsilon\xi) \int_{\mathbb{R}^n} |z^{\varepsilon\xi,\sigma}|^{p+1}. \end{aligned}$$

From this it follows readily that we can find some $c_0 > 0$ such that for all $\varepsilon > 0$ small, all $|\xi| \leq \bar{\xi}$ and all $\sigma \in [0, 2\pi]$ it results

$$D^2 F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}) < c_0 < 0. \quad (18)$$

Recalling (10), we find

$$\begin{aligned} \langle L_{\varepsilon,\sigma,\xi} z^{\varepsilon\xi,\sigma} \mid z^{\varepsilon\xi,\sigma} \rangle &= D^2 F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}) \\ &\quad + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)] |z^{\varepsilon\xi,\sigma}|^2 - \int_{\mathbb{R}^n} [K(\varepsilon x) - K(\varepsilon\xi)] |z^{\varepsilon\xi,\sigma}|^{p+1} \\ &\quad + \int_{\mathbb{R}^n} \left(\left| \left(\frac{\nabla}{i} - A(\varepsilon x) \right) z^{\varepsilon\xi,\sigma} \right|^2 - \left| \left(\frac{\nabla}{i} - A(\varepsilon\xi) \right) z^{\varepsilon\xi,\sigma} \right|^2 \right). \end{aligned}$$

It follows that

$$\begin{aligned} \langle L_{\varepsilon,\sigma,\xi} z^{\varepsilon\xi,\sigma} \mid z^{\varepsilon\xi,\sigma} \rangle &\leq D^2 F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}) \\ &\quad + c_1 \varepsilon |\nabla V(\varepsilon\xi)| + c_2 \varepsilon |\nabla K(\varepsilon\xi)| + c_3 \varepsilon |J_A(\varepsilon\xi)| + c_4 \varepsilon^2. \end{aligned} \quad (19)$$

Hence (16) follows. The proof of (17) is more involved. We first prove the following claim.

Claim. There results

$$D^2 F^{\varepsilon\xi, \sigma}(z^{\varepsilon\xi, \sigma})(\phi, \phi) \geq c_1 \|\phi\|^2 \quad \forall \phi \perp \mathcal{V}. \quad (20)$$

Recall that the complex ground state U_c introduced in (6) is a critical point of mountain-pass type for the corresponding energy functional $J : E \rightarrow \mathbb{R}$ defined by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + |u|^2 + V(\varepsilon\xi)|u|^2) - \frac{1}{p+1} \int_{\mathbb{R}^n} K(\varepsilon\xi)|u|^{p+1}. \quad (21)$$

Let

$$\mathcal{M} = \left\{ u \in E : \int_{\mathbb{R}^n} (|\nabla u|^2 + |u|^2 + V(\varepsilon\xi)|u|^2) = \int_{\mathbb{R}^n} |u|^{p+1} \right\}$$

be the Nehari manifold of J , which has codimension one. Let

$$\mathcal{N} = \left\{ u \in E : \int_{\mathbb{R}^n} \left(\left| \left(\frac{\nabla}{i} - A(\varepsilon\xi) \right) u \right|^2 + |u|^2 + V(\varepsilon\xi)|u|^2 \right) = \int_{\mathbb{R}^n} |u|^{p+1} \right\}$$

be the Nehari manifold of $F^{\varepsilon\xi, \sigma}$. One checks readily that $\text{codim } \mathcal{N} = 1$. Recall [21] that U_c is, up to multiplication by a constant phase, the *unique* minimum of J restricted to \mathcal{M} . Now, for every $u \in \mathcal{M}$, the function $x \mapsto e^{iA(\varepsilon\xi) \cdot x} u(x)$ lies in \mathcal{N} , and vice versa. Moreover

$$J(u) = F^{\varepsilon\xi, \sigma}(e^{iA(\varepsilon\xi) \cdot x} u).$$

This immediately implies that $\min_{\mathcal{N}} F^{\varepsilon\xi, \sigma}$ is achieved at a point which differs from $e^{iA(\varepsilon\xi) \cdot x} U_c(x)$ at most for a constant phase. In other words, $z^{\varepsilon\xi, \sigma}$ is a critical point for $F^{\varepsilon\xi, \sigma}$ of mountain-pass type, and the claim follows by standard results (see [7]).

Let $R \gg 1$ and consider a radial smooth function $\chi_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\chi_1(x) = 1, \quad \text{for } |x| \leq R; \quad \chi_1(x) = 0, \quad \text{for } |x| \geq 2R; \quad (22)$$

$$|\nabla \chi_1(x)| \leq \frac{2}{R}, \quad \text{for } R \leq |x| \leq 2R. \quad (23)$$

We also set $\chi_2(x) = 1 - \chi_1(x)$. Given ϕ let us consider the functions

$$\phi_i(x) = \chi_i(x - \xi)\phi(x), \quad i = 1, 2.$$

A straightforward computation yields:

$$\int_{\mathbb{R}^n} |\phi|^2 = \int_{\mathbb{R}^n} |\phi_1|^2 + \int_{\mathbb{R}^n} |\phi_2|^2 + 2 \operatorname{Re} \int_{\mathbb{R}^n} \phi_1 \bar{\phi}_2,$$

$$\int_{\mathbb{R}^n} |\nabla \phi|^2 = \int_{\mathbb{R}^n} |\nabla \phi_1|^2 + \int_{\mathbb{R}^n} |\nabla \phi_2|^2 + 2 \operatorname{Re} \int_{\mathbb{R}^n} \nabla \phi_1 \cdot \overline{\nabla \phi_2},$$

and hence

$$\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2 \operatorname{Re} \int_{\mathbb{R}^n} [\phi_1 \overline{\phi_2} + \nabla \phi_1 \cdot \overline{\nabla \phi_2}].$$

Letting I denote the last integral, one immediately finds:

$$\begin{aligned} I &= \underbrace{\int_{\mathbb{R}^n} \chi_1 \chi_2 (\phi^2 + |\nabla \phi|^2)}_{I_\phi} + \underbrace{\int_{\mathbb{R}^n} \phi^2 \nabla \chi_1 \cdot \nabla \chi_2}_{I'} \\ &\quad + \underbrace{\int_{\mathbb{R}^n} (\phi_2 \nabla \chi_1 \cdot \overline{\nabla \phi} + \overline{\phi_1} \nabla \phi \cdot \nabla \chi_2)}_{I''}. \end{aligned}$$

Due to the definition of χ , the two integrals I' and I'' reduce to integrals from R and $2R$, and thus they are $o_R(1)\|\phi\|^2$, where $o_R(1)$ is a function which tends to 0, as $R \rightarrow +\infty$. As a consequence we have that

$$\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2I_\phi + o_R(1)\|\phi\|^2. \quad (24)$$

After these preliminaries, let us evaluate the three terms in the equation below:

$$(L_{\varepsilon, \sigma, \xi} \phi | \phi) = \underbrace{(L_{\varepsilon, \sigma, \xi} \phi_1 | \phi_1)}_{\alpha_1} + \underbrace{(L_{\varepsilon, \sigma, \xi} \phi_2 | \phi_2)}_{\alpha_2} + 2 \underbrace{(L_{\varepsilon, \sigma, \xi} \phi_1 | \phi_2)}_{\alpha_3}.$$

One has:

$$\begin{aligned} \alpha_1 &= \langle L_{\varepsilon, \sigma, \xi} \phi_1 | \phi_1 \rangle = D^2 F^{\varepsilon \xi, \sigma}(z^{\varepsilon \xi, \sigma})(\phi_1, \phi_1) + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)] |\phi_1|^2 \\ &\quad - \int_{\mathbb{R}^n} [K(\varepsilon x) - K(\varepsilon \xi)] |\phi_1|^{p+1} \\ &\quad + \int_{\mathbb{R}^n} \left(\left| \left(\frac{\nabla}{i} - A(\varepsilon x) \right) \phi_1 \right|^2 - \left| \left(\frac{\nabla}{i} - A(\varepsilon \xi) \right) \phi_1 \right|^2 \right). \end{aligned}$$

In order to use (20), we introduce the function $\phi_1^* = \phi_1 - \psi$, where ψ is the projection of ϕ_1 onto \mathcal{V} :

$$\begin{aligned} \psi &= (\phi_1 | z^{\varepsilon \xi, \sigma}) z^{\varepsilon \xi, \sigma} \|z^{\varepsilon \xi, \sigma}\|^{-2} + (\phi_1 | i z^{\varepsilon \xi, \sigma}) i z^{\varepsilon \xi, \sigma} \|z^{\varepsilon \xi, \sigma}\|^{-2} \\ &\quad + \sum (\phi_1 | \partial_{x_i} z^{\varepsilon \xi, \sigma}) \partial_{x_i} z^{\varepsilon \xi, \sigma} \|\partial_{x_i} z^{\varepsilon \xi, \sigma}\|^{-2}. \end{aligned}$$

Then we have:

$$\begin{aligned}
D^2 F^{\varepsilon\xi}[\phi_1, \phi_1] \\
= D^2 F^{\varepsilon\xi}[\phi_1^*, \phi_1^*] + D^2 F^{\varepsilon\xi}[\psi, \psi] + 2 \operatorname{Re} D^2 F^{\varepsilon\xi}[\phi_1^*, \psi].
\end{aligned} \quad (25)$$

Since $z^{\varepsilon\xi, \sigma}$ is orthogonal to $\partial_{x_i} z^{\varepsilon\xi, \sigma}$, $i = 1, \dots, n$, then one readily checks that $\phi_1^* \perp \mathcal{V}$ and hence (20) implies

$$D^2 F^{\varepsilon\xi}[\phi_1^*, \phi_1^*] \geq c_1 \|\phi_1^*\|^2. \quad (26)$$

On the other side, since $(\phi|z^{\varepsilon\xi, \sigma}) = 0$ it follows:

$$\begin{aligned}
(\phi_1|z^{\varepsilon\xi, \sigma}) &= (\phi|z^{\varepsilon\xi, \sigma}) - (\phi_2|z^{\varepsilon\xi, \sigma}) = -(\phi_2|z^{\varepsilon\xi, \sigma}) \\
&= -\operatorname{Re} \int_{\mathbb{R}^n} \phi_2 z^{\varepsilon\xi, \sigma} - \operatorname{Re} \int_{\mathbb{R}^n} \nabla z^{\varepsilon\xi, \sigma} \cdot \nabla \phi_2 \\
&= -\operatorname{Re} \int_{\mathbb{R}^n} \chi_2(y) z(y) \phi(y + \xi) dy \\
&\quad - \operatorname{Re} \int_{\mathbb{R}^n} \nabla z(y) \cdot \nabla \chi_2(y) \phi(y + \xi) dy.
\end{aligned}$$

Since $\chi_2(x) = 0$ for all $|x| < R$, and since $z(x) \rightarrow 0$ as $|x| = R \rightarrow \infty$, we infer $(\phi_1|z^{\varepsilon\xi, \sigma}) = o_R(1)\|\phi\|$. Similarly one shows that $(\phi_1|\partial_x z^{\varepsilon\xi, \sigma}) = o_R(1)\|\phi\|$ and it follows that

$$\|\psi\| = o_R(1)\|\phi\|. \quad (27)$$

We are now in position to estimate the last two terms in Eq. (25). Actually, using Lemma 3.1 we get

$$\begin{aligned}
D^2 F^{\varepsilon\xi}[\psi, \psi] &\geq C\|\psi\|^2 + V(\varepsilon\xi) \int_{\mathbb{R}^n} \psi^2 \\
&\quad - K(\varepsilon\xi) \left[\operatorname{Re}(p-1) \int_{\mathbb{R}^n} |z^{\varepsilon\xi, \sigma}|^{p-3} \operatorname{Re}(z^{\varepsilon\xi, \sigma} \bar{\psi}) z^{\varepsilon\xi, \sigma} \bar{\psi} \right. \\
&\quad \left. + \int_{\mathbb{R}^n} |z^{\varepsilon\xi, \sigma}|^{p-1} |\psi|^2 \right] = o_R(1)\|\phi\|^2.
\end{aligned} \quad (28)$$

The same arguments readily imply

$$\operatorname{Re} D^2 F^{\varepsilon\xi}[\phi_1^*, \psi] = o_R(1)\|\phi\|^2. \quad (29)$$

Putting together (26), (28) and (29) we infer

$$D^2 F^{\varepsilon\xi}[\phi_1, \phi_1] \geq C\|\phi_1\|^2 + o_R(1)\|\phi\|^2. \quad (30)$$

Using arguments already carried out before, one has

$$\begin{aligned}
\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)| \phi_1^2 &\leq \varepsilon c_2 \int_{\mathbb{R}^n} |x - \xi| \chi_1^2(x - \xi) \phi^2(x) \\
&\leq \varepsilon c_3 \int_{\mathbb{R}^n} |y| \chi_1^2(y) \phi^2(y + \xi) dy \\
&\leq \varepsilon c_4 R \|\phi\|^2,
\end{aligned}$$

and similarly for the terms containing K . This and (30) yield

$$\alpha_1 = (L_{\varepsilon, \sigma, \xi} \phi_1 | \phi_1) \geq c_5 \|\phi_1\|^2 - \varepsilon c_4 R \|\phi\|^2 + o_R(1) \|\phi\|^2. \quad (31)$$

Let us now estimate α_2 . One finds

$$\alpha_2 = (L_{\varepsilon, \sigma, \xi} \phi_2 | \phi_2) \geq c_6 \|\phi_2\|^2 + o_R(1) \|\phi\|^2. \quad (32)$$

In a quite similar way one shows that

$$\alpha_3 \geq c_7 I_\phi + o_R(1) \|\phi\|^2. \quad (33)$$

Finally, (31), (32), (33) and the fact that $I_\phi \geq 0$, yield

$$\begin{aligned}
(L_{\varepsilon, \sigma, \xi} \phi | \phi) &= \alpha_1 + \alpha_2 + 2\alpha_3 \\
&\geq c_8 [\|\phi_1\|^2 + \|\phi_2\|^2 + 2I_\phi] - c_9 R \varepsilon \|\phi\|^2 + o_R(1) \|\phi\|^2.
\end{aligned}$$

Recalling (24) we infer that

$$(L_{\varepsilon, \sigma, \xi} \phi | \phi) \geq c_{10} \|\phi\|^2 - c_9 R \varepsilon \|\phi\|^2 + o_R(1) \|\phi\|^2.$$

Taking $R = \varepsilon^{-1/2}$, and choosing ε small, Eq. (17) follows. This completes the proof. \square

4. The finite-dimensional reduction

In this section we will show that the existence of critical points of f_ε can be reduced to the search of critical points of an auxiliary finite-dimensional functional. The proof will be carried out in two subsections dealing, respectively, with a Liapunov–Schmidt reduction, and with the behavior of the auxiliary finite-dimensional functional.

4.1. A Liapunov–Schmidt type reduction

The main result of this section is the following lemma.

Lemma 4.1. *For $\varepsilon > 0$ small, $|\xi| \leq \bar{\xi}$ and $\sigma \in [0, 2\pi]$, there exists a unique $w = w(\varepsilon, \sigma, \xi) \in (T_{z^{\varepsilon\xi, \sigma}} Z^\varepsilon)^\perp$ such that $\nabla f_\varepsilon(z^{\varepsilon\xi, \sigma} + w) \in T_{z^{\varepsilon\xi, \sigma}} Z^\varepsilon$. Such a $w(\varepsilon, \sigma, \xi)$ is of class C^2 , respectively $C^{1, p-1}$, with respect to ξ , provided that*

$p \geq 2$, respectively $1 < p < 2$. Moreover, the functional $\Phi_\varepsilon(\sigma, \xi) = f_\varepsilon(z^{\varepsilon\xi, \sigma} + w(\varepsilon, \sigma, \xi))$ has the same regularity as w and satisfies:

$$\nabla \Phi_\varepsilon(\sigma_0, \xi_0) = 0 \iff \nabla f_\varepsilon(z_{\xi_0} + w(\varepsilon, \sigma_0, \xi_0)) = 0.$$

Proof. Let $P = P_{\varepsilon\xi, \sigma}$ denote the projection onto $(T_{z^{\varepsilon\xi, \sigma}} Z^\varepsilon)^\perp$. We want to find a solution $w \in (T_{z^{\varepsilon\xi, \sigma}} Z)^\perp$ of the equation $P \nabla f_\varepsilon(z^{\varepsilon\xi, \sigma} + w) = 0$. One has that $\nabla f_\varepsilon(z + w) = \nabla f_\varepsilon(z) + D^2 f_\varepsilon(z)[w] + R(z, w)$ with $\|R(z, w)\| = o(\|w\|)$, uniformly with respect to $z = z^{\varepsilon\xi, \sigma}$, for $|\xi| \leq \bar{\xi}$. Using the notation introduced in the previous section, we are led to the equation:

$$L_{\varepsilon, \sigma, \xi} w + P \nabla f_\varepsilon(z) + P R(z, w) = 0.$$

According to Lemma 3.2, this is equivalent to

$$w = N_{\varepsilon, \xi, \sigma}(w), \quad \text{where } N_{\varepsilon, \xi, \sigma}(w) = -L_{\varepsilon, \sigma, \xi}^{-1}(P \nabla f_\varepsilon(z) + P R(z, w)).$$

From Lemma 2.1 it follows that

$$\begin{aligned} \|N_{\varepsilon, \xi, \sigma}(w)\| &\leq c_1(\varepsilon |\nabla V(\varepsilon\xi)| + \varepsilon |\nabla K(\varepsilon\xi)| + \varepsilon |J_A(\varepsilon\xi)| + \varepsilon^2) \\ &\quad + o(\|w\|). \end{aligned} \quad (34)$$

Then one readily checks that $N_{\varepsilon, \xi, \sigma}$ is a contraction on some ball in $(T_{z^{\varepsilon\xi, \sigma}} Z^\varepsilon)^\perp$ provided that $\varepsilon > 0$ is small enough and $|\xi| \leq \bar{\xi}$. Then there exists a unique w such that $w = N_{\varepsilon, \xi, \sigma}(w)$. Let us point out that we cannot use the Implicit Function Theorem to find $w(\varepsilon, \xi, \sigma)$, because the map $(\varepsilon, u) \mapsto P \nabla f_\varepsilon(u)$ fails to be C^2 . However, fixed $\varepsilon > 0$ small, we can apply the Implicit Function Theorem to the map $(\xi, \sigma, w) \mapsto P \nabla f_\varepsilon(z^{\varepsilon\xi, \sigma} + w)$. Then, in particular, the function $w(\varepsilon, \xi, \sigma)$ turns out to be of class C^1 with respect to ξ and σ . Finally, it is a standard argument, see [1,2], to check that the critical points of $\Phi_\varepsilon(\xi, \sigma) = f_\varepsilon(z + w)$ give rise to critical points of f_ε . \square

Remark 4.1. Since $f_\varepsilon(z^{\varepsilon\xi, \sigma})$ is actually independent of σ , the implicit function w is constant with respect to that variable. As a result, there exists a functional $\Psi_\varepsilon: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\Phi_\varepsilon(\sigma, \xi) = \Psi_\varepsilon(\xi), \quad \forall \sigma \in [0, 2\pi], \quad \forall \xi \in \mathbb{R}^n.$$

In the sequel, we will omit the dependence of w on σ , even it is defined over $S^1 \times \mathbb{R}^n$.

Remark 4.2. From (34) it immediately follows that:

$$\|w\| \leq C(\varepsilon |\nabla V(\varepsilon\xi)| + \varepsilon |\nabla K(\varepsilon\xi)| + \varepsilon |J_A(\varepsilon\xi)| + \varepsilon^2), \quad (35)$$

where $C > 0$.

The following result can be proved by adapting the same argument as in [3].

Lemma 4.2. *One has that:*

$$\|\nabla_{\xi} w\| \leq c \left(\varepsilon |\nabla V(\varepsilon \xi)| + \varepsilon |\nabla K(\varepsilon \xi)| + \varepsilon |J_A(\varepsilon \xi)| + O(\varepsilon^2) \right)^{\gamma}, \quad (36)$$

where $\gamma = \min\{1, p-1\}$ and $c > 0$ is some constant.

4.2. The finite-dimensional functional

The purpose of this subsection is to give an explicit form to the finite-dimensional functional $\Phi_{\varepsilon}(\sigma, \xi) = \Psi_{\varepsilon}(\xi) = f_{\varepsilon}(z^{\varepsilon \xi, \sigma} + w(\varepsilon, \xi))$.

Recall the precise definition of $z^{\varepsilon \xi, \sigma}$ given in (9). For brevity, we set in the sequel $z = z^{\varepsilon \xi, \sigma}$ and $w = w(\varepsilon, \xi)$.

Since z satisfies (5), we easily find the following relations:

$$\int_{\mathbb{R}^n} \left| \left(\frac{\nabla}{i} - A(\varepsilon \xi) \right) z \right|^2 + |z|^2 + V(\varepsilon \xi) |z|^2 = \int_{\mathbb{R}^n} K(\varepsilon \xi) |z|^{p+1}, \quad (37)$$

$$\begin{aligned} & \operatorname{Re} \int_{\mathbb{R}^n} \left(\frac{\nabla}{i} - A(\varepsilon \xi) \right) z \cdot \overline{\left(\frac{\nabla}{i} - A(\varepsilon \xi) \right) w} + \operatorname{Re} \int_{\mathbb{R}^n} z \bar{w} + \operatorname{Re} \int_{\mathbb{R}^n} V(\varepsilon \xi) z \bar{w} \\ &= \operatorname{Re} \int_{\mathbb{R}^n} K(\varepsilon \xi) |z|^{p-1} z \bar{w}. \end{aligned} \quad (38)$$

Hence we get

$$\begin{aligned} \Phi_{\varepsilon}(\sigma, \xi) &= f_{\varepsilon}(z^{\varepsilon \xi, \sigma} + w(\varepsilon, \sigma, \xi)) \\ &= K(\varepsilon \xi) \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^n} |z|^{p+1} + \frac{1}{2} \int_{\mathbb{R}^n} |A(\varepsilon \xi) - A(\varepsilon x)|^2 z^2 \\ &\quad + \operatorname{Re} \int_{\mathbb{R}^n} (A(\varepsilon \xi) - A(\varepsilon x)) z \cdot (A(\varepsilon \xi) - A(\varepsilon x)) \bar{w} \\ &\quad + \varepsilon \operatorname{Re} \int_{\mathbb{R}^n} \frac{1}{i} z \bar{w} \operatorname{div} A(\varepsilon x) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} \left| \left(\frac{\nabla}{i} - A(\varepsilon x) \right) w \right|^2 + \operatorname{Re} \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)] z \bar{w} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)] |w|^2 + \frac{1}{2} \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)] z^2 \\ &\quad + \frac{1}{2} V(\varepsilon \xi) \int_{\mathbb{R}^n} |w|^2 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{p+1} \operatorname{Re} \int_{\mathbb{R}^n} K(\varepsilon x) (|z+w|^{p+1} - |z|^{p+1} - (p+1)|z|^{p-1}z\bar{w}) \\
& + \operatorname{Re} K(\varepsilon \xi) \int_{\mathbb{R}^n} |z|^{p-1}z\bar{w} + O(\varepsilon^2).
\end{aligned} \tag{39}$$

Here we have used the estimate

$$\begin{aligned}
& \int_{\mathbb{R}^n} \left(\frac{1}{2} K(\varepsilon x) - \frac{1}{p+1} K(\varepsilon \xi) \right) |z|^{p+1} \\
& = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^n} K(\varepsilon \xi) |z|^{p+1} + O(\varepsilon^2),
\end{aligned}$$

which follows from the boundedness of K'' . Since we know that

$$\begin{aligned}
\alpha(\varepsilon \xi) &= \left(\frac{1 + V(\varepsilon \xi)}{K(\varepsilon \xi)} \right)^{1/(p-1)}, \\
\beta(\varepsilon \xi) &= (1 + V(\varepsilon \xi))^{1/2},
\end{aligned}$$

we get immediately

$$\int_{\mathbb{R}^n} |z^{\varepsilon \xi, \sigma}|^{p+1} = C_0 \Lambda(\varepsilon \xi) [K(\varepsilon \xi)]^{-1}, \tag{40}$$

where we define the auxiliary function

$$\Lambda(x) = \frac{(1 + V(x))^\theta}{K(x)^{2/(p-1)}}, \quad \theta = \frac{p+1}{p-1} - \frac{n}{2}, \tag{41}$$

and $C_0 = \|U\|_{L^2}$. Now one can estimate the various terms in (39) by means of (35) and (36), to prove that

$$\Phi_\varepsilon(\sigma, \xi) = \Psi_\varepsilon(\xi) = C_1 \Lambda(\varepsilon \xi) + O(\varepsilon). \tag{42}$$

Similarly,

$$\nabla \Psi_\varepsilon(\xi) = C_1 \nabla \Lambda(\varepsilon \xi) + \varepsilon^{1+\gamma} O(1), \tag{43}$$

where $C_1 = (\frac{1}{2} - \frac{1}{p+1})C_0$. We omit the details, which can be deduced without effort from [3].

5. Statement and proof of the main results

In this section we exploit the finite-dimensional reduction performed in the previous section to find existence and multiple solutions of (NLS). Recalling

Lemma 4.1, we have to look for critical points of Φ_ε as a function of the variables $(\sigma, \xi) \in [0, 2\pi] \times \mathbb{R}^n$ (or, equivalently, $(\eta, \xi) \in S^1 \times \mathbb{R}^n$).

In what follows, we use the following notation: given a set $\Omega \subset \mathbb{R}^n$ and a number $\rho > 0$,

$$\Omega_\rho \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < \rho\}.$$

We start with the following result, which deals with local extrema.

Theorem 5.1. *Suppose that (K1), (V1) and (A1) hold. Assume moreover that there is a compact set $M \subset \mathbb{R}^n$ over which Λ achieves an isolated strict local minimum with value a . By this we mean that for some $\delta > 0$,*

$$b \stackrel{\text{def}}{=} \inf_{x \in \partial M_\delta} \Lambda(x) > a. \quad (44)$$

Then there exists $\varepsilon_\delta > 0$ such that (S_ε) has at least $\text{cat}(M, M_\delta)$ (orbits of) solutions concentrating near M_δ , for all $0 < \varepsilon < \varepsilon_\delta$.

Conversely, assume that K is a compact set \mathbb{R}^n over which Λ achieves an isolated strict local maximum with value b , namely for some $\gamma > 0$,

$$a \stackrel{\text{def}}{=} \inf_{x \in \partial K_\gamma} \Lambda(x) < b. \quad (45)$$

Then there exists $\varepsilon_\gamma > 0$ such that (S_ε) has at least $\text{cat}(K, K_\gamma)$ (orbits of) solutions concentrating near K_γ , for all $0 < \varepsilon < \varepsilon_\gamma$.

Proof. As in the previous theorem, one has $\Phi_\varepsilon(\eta, \xi) = \Psi_\varepsilon(\xi)$. Now choose $\bar{\xi} > 0$ in such a way that $M_\delta \subset \{x \in \mathbb{R}^n \mid |x| < \bar{\xi}\}$. Define again $\bar{\Lambda}$ as in the proof of Theorem 5.3. Let

$$\begin{aligned} N^\varepsilon &= \{\xi \in \mathbb{R}^n \mid \varepsilon\xi \in M\}, \\ N_\delta^\varepsilon &= \{\xi \in \mathbb{R}^n \mid \varepsilon\xi \in M_\delta\}, \\ \Theta^\varepsilon &= \left\{ \xi \in \mathbb{R}^n \mid \Psi_\varepsilon(\xi) \leq C_1 \frac{a+b}{2} \right\}. \end{aligned}$$

From (42) we get some $\varepsilon_\delta > 0$ such that

$$N^\varepsilon \subset \Theta^\varepsilon \subset N_\delta^\varepsilon, \quad (46)$$

for all $0 < \varepsilon < \varepsilon_\delta$. To apply standard category theory, we need to prove that Θ^ε is compact. To this end, as can be readily checked, it suffices to prove that Θ^ε cannot touch $\partial N_\delta^\varepsilon$. But if $\varepsilon\xi \in \partial M$, one has $\bar{\Lambda}(\varepsilon\xi) \geq b$ by the very definition of δ , and so

$$\Psi_\varepsilon(\xi) \geq C_1 \bar{\Lambda}(\varepsilon\xi) + o_\varepsilon(1) \geq C_1 b + o_\varepsilon(1).$$

On the other hand, for all $\xi \in \Theta^\varepsilon$ one has also $\Psi_\varepsilon(\xi) \leq C_1 \frac{a+b}{2}$. We can conclude from (46) and elementary properties of the Lusternik–Schnirel'man category that Ψ_ε has at least

$$\text{cat}(\Theta^\varepsilon, \Theta^\varepsilon) \geq \text{cat}(N^\varepsilon, N_\delta^\varepsilon) = \text{cat}(N, M_\delta)$$

critical points in Θ^ε , which correspond to at least $\text{cat}(M, M_\delta)$ orbits of solutions to (S_ε) . Now, let $(\eta^*, \xi^*) \in S^1 \times M_\delta$ a critical point of Φ_ε . Hence this point (η^*, ξ^*) localizes a solution $u_{\varepsilon, \eta^*, \xi^*}(x) = z^{\varepsilon \xi^*, \eta^*}(x) + w(\varepsilon, \eta^*, \xi^*)$ of (S_ε) . Recalling the change of variable which allowed us to pass from (NLS) to (S_ε) , we find that

$$u_{\varepsilon, \eta^*, \xi^*}(x) \approx z^{\varepsilon \xi^*, \eta^*} \left(\frac{x - \xi^*}{\varepsilon} \right)$$

solves (NLS). The concentration statement follows from standard arguments ([2,3]). The proof of the second part of Theorem 5.3 follows with analogous arguments. \square

Theorem 1.1 in the Introduction is an immediate corollary of the previous one when x_0 is either a nondegenerate local maximum or minimum for Λ . We remark that the case in which Λ has a maximum cannot be handled using a direct variational approach and the arguments in [8] cannot be applied.

To treat the general case, we need some more work. In order to present our main result, we need to introduce some topological concepts.

Given a set $M \subset \mathbb{R}^n$, the *cup long* of M is by definition

$$\ell(M) = 1 + \sup \{k \in \mathbb{N} \mid (\exists \alpha_1, \dots, \alpha_n \in \check{H}^*(M) \setminus \{1\}) (\alpha_1 \cup \dots \cup \alpha_k \neq 0)\}.$$

If no such classes exists, we set $\ell(M) = 1$. Here $\check{H}^*(M)$ is the Alexander cohomology of M with real coefficients, and \cup denotes the cup product. It is well known that $\ell(S^{n-1}) = \text{cat}(S^{n-1}) = 2$, and $\ell(T^n) = \text{cat}(T^n) = n + 1$, where T^n is the standard n -dimensional torus. But in general, one has $\ell(M) \leq \text{cat}(M)$.

The following definition dates back to Bott ([5]).

Definition 5.1. We say that M is non-degenerate for a C^2 function $I : \mathbb{R}^N \rightarrow \mathbb{R}$ if M consists of Morse theoretically non-degenerate critical points for the restriction $I|_{M^\perp}$.

To prove our existence result, we need the next theorem, which is a slightly modified statement of Theorem 6.4 in Chapter II of [7].

Theorem 5.2. Let $I \in C^1(V)$ and $J \in C^2(V)$ be two functionals defined on the Riemannian manifold V , and let $\Sigma \subset V$ be a smooth, compact, non-degenerate manifold of critical points of J . Denote by \mathcal{U} a neighborhood of Σ .

If $\|I - J\|_{C^1(\mathcal{U})}$ is small enough, then the functional I has at least $\ell(\Sigma)$ critical points contained in \mathcal{U} .

We only remark that Theorem 5.2 can also be proved in the framework of Conley theory ([12]).

We are now ready to prove an existence and multiplicity result for (NLS).

Theorem 5.3. *Let (V1), (K1) and (A1) hold. If the auxiliary function Λ has a smooth, compact, non-degenerate manifold of critical points M , then for $\varepsilon > 0$ small, the problem (S_ε) has at least $\ell(M)$ (orbits of) solutions concentrating near points of M .*

Proof. By Remark 4.1, we have to find critical points of $\Psi_\varepsilon = \Psi_\varepsilon(\xi)$. Since M is compact, we can choose $\bar{\xi} > 0$ so that $|x| < \bar{\xi}$ for all points $x \in M$. From this moment, $\bar{\xi}$ is kept fixed. Form $\{\eta^*\} \times M$ is obviously a non-degenerate critical manifold. We set now $V = \mathbb{R}^n$, $J = \Lambda$, $\Sigma = M$, and $I(\xi) = \Psi_\varepsilon(\eta, \xi/\varepsilon)$. Select $\delta > 0$ so that $M_\delta \subset \{x: |x| < \bar{\xi}\}$, and no critical points of Λ are in M_δ , except for those of M . Set $\mathcal{U} = M_\delta$. From (42) and (43) it follows that I is close to J in $C^1(\bar{\mathcal{U}})$ when ε is very small. We can apply Theorem 5.2 to find at least $\ell(M)$ critical points $\{\xi_1, \dots, \xi_{\ell(M)}\}$ for Ψ_ε , provided ε is small enough. Hence the orbits $S^1 \times \{\xi_1\}, \dots, S^1 \times \{\xi_{\ell(M)}\}$ consist of critical points for Φ_ε which produce solutions of (S_ε) . The concentration statement follows as in [3]. \square

Remark 5.1. We point out that Theorem 1.1, Theorem 5.1 and Theorem 5.3 hold for problems involving more general nonlinearities $g(x, u)$ satisfying the same assumptions in [16] (see also Remark 5.4 in [3]). For our approach, we need the uniqueness of the radial solution z of the corresponding scalar equation

$$-\Delta u + u + V(\varepsilon\xi)u = g(\varepsilon\xi, u), \quad u > 0, u \in W^{1,2}(\mathbb{R}^n). \quad (47)$$

Let us also remark that in [8] the class of nonlinearities handled does not require that Eq. (47) has a unique solution.

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